On the Properties of Nonlinear Integral Equations That Arise in the Theory of Dynamical Systems

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This paper reports on some results concerning the properties of integral equations that govern the behavior of a large class of control systems or electrical networks containing linear time-invariant elements and an arbitrary finite number of nonlinear time-varying elements.

In particular, for networks containing linear time-invariant elements and an arbitrary finite number of positive-slope nonlinear resistors, it is proved, under reasonable conditions, that the response to a periodic excitation applied at $t = 0$ is ultimately periodic with the same period as the excitation, regardless of the initial state of the network.

I. NOTATION AND DEFINITIONS

Let M denote an arbitrary matrix. We shall denote by M', M^*, and M^{-1}, respectively, the transpose, the complex-conjugate transpose, and the inverse of M. The positive square-root of the largest eigenvalue of M^*M is denoted by $\Lambda\{M\}$, and 1_N denotes the identity matrix of order N.

The set of real, measurable N-vector-valued functions of the real variable t defined on $(-\infty, \infty)$ $[[0, \infty)]$ is denoted by $\mathcal{K}_N$ $[\mathcal{K}_{N+}]$, and

$$\mathfrak{L}_{2N} = \left\{ f \mid f \varepsilon \mathcal{K}_N, \int_{-\infty}^{\infty} f'f \, dt < \infty \right\}$$

$$\mathfrak{L}_{2N+} = \left\{ f \mid f \varepsilon \mathcal{K}_{N+}, \int_{0}^{\infty} f'f \, dt < \infty \right\}.$$

The norm of $f = (f_1, f_2, \cdots, f_N)' \varepsilon \mathfrak{L}_{2N}$ $[\mathfrak{L}_{2N+}]$ is denoted by

$$\| f \| \; [\| f \|_+];$$

it is defined by

$$\| f \|^2 = \int_{-\infty}^{\infty} f' f \, dt \qquad \left[\| f \|_+^2 = \int_0^{\infty} f' f \, dt \right]$$

and the norm of a linear transformation $\mathbf{T}$ defined on $\mathcal{L}_{2N}$ $[\mathcal{L}_{2N+}]$ is denoted by $\| \mathbf{T} \|$ $[\| \mathbf{T} \|_+]$.

Let $y \, \varepsilon \, (0, \infty)$, and, if $f \, \varepsilon \, \mathcal{H}_N$, let

$$\begin{aligned} f_y &= f \quad \text{for} \quad |t| \leqq y \\ &= 0 \quad \text{for} \quad |t| > y; \end{aligned}$$

if $f \, \varepsilon \, \mathcal{H}_{N+}$, let

$$\begin{aligned} f_y &= f \quad \text{for} \quad t \, \varepsilon \, [0,y] \\ &= 0 \quad \text{for} \quad t > y. \end{aligned}$$

The sets $\mathcal{E}_N$ and $\mathcal{E}_{N+}$ are defined as follows

$$\mathcal{E}_N = \{ f \mid f \, \varepsilon \, \mathcal{H}_N, \quad f_y \, \varepsilon \, \mathcal{L}_{2N} \text{ for } 0 < y < \infty \}$$

$$\mathcal{E}_{N+} = \{ f \mid f \, \varepsilon \, \mathcal{H}_{N+}, \quad f_y \, \varepsilon \, \mathcal{L}_{2N+} \text{ for } 0 < y < \infty \}.$$

With $\mathcal{B}$ the set of N-vector-valued functions of t which have the property that each component is uniformly bounded on its domain of definition, let

$$\mathcal{L}_{\infty N} = \mathcal{B} \cap \mathcal{H}_N, \quad \text{and} \quad \mathcal{L}_{\infty N+} = \mathcal{B} \cap \mathcal{H}_{N+}.$$

Let T be a real positive constant and let

$$\mathcal{K}_N = \left\{ f \mid f \, \varepsilon \, \mathcal{H}_N, \quad f(t) = f(t + T) \quad \text{for all } t, \int_0^T f' f \, dt < \infty \right\}.$$

Throughout the paper, k denotes a measurable, real $N \times N$ matrix-valued function of t defined on $(-\infty, \infty)$, with elements $\{k_{mn}\}$ such that

$$\int_{-\infty}^{\infty} | k_{mn}(t) | \, dt < \infty \qquad (m,n = 1, 2, \cdots, N),$$

and $\psi[f(t),t]$, with $f \, \varepsilon \, \mathcal{H}_N$ or $f \, \varepsilon \, \mathcal{H}_{N+}$, denotes the N vector

$$(\psi_1[f_1(t),t], \psi_2[f_2(t),t], \cdots, \psi_N[f_N(t),t])'$$

where $\psi_1(w,t), \psi_2(w,t), \cdots, \psi_N(w,t)$ are real-valued functions of the real variables w and t for $-\infty < w < \infty$ and $-\infty < t < \infty$ such that

(*i*) there exist real numbers α and β with the property that

$$\alpha \leqq \frac{\psi_n(w_1, t) - \psi_n(w_2, t)}{w_1 - w_2} \leqq \beta \qquad (n = 1, 2, \cdots, N)$$

for all $t \varepsilon (-\infty, \infty)$ and all real w_1 and w_2 such that $w_1 \neq w_2$, and

(*ii*) $\psi_n[w(t),t]$ is a measurable function of t whenever $w(t)$ is measurable ($n = 1, 2, \cdots, N$).

The symbol s denotes a scalar complex variable with $\sigma = \mathrm{Re}[s]$ and $\omega = \mathrm{Im}[s]$.

II. INTRODUCTION

Equations of the form

$$g(t) = f(t) + \int_0^t k(t - \tau)\psi[f(\tau),\tau]d\tau, \qquad 0 \leqq t < \infty \tag{1}$$

in which $f \varepsilon \mathcal{E}_{N+}$ and $g \varepsilon \mathcal{L}_{\infty N+}$, are frequently encountered in the study of physical systems containing linear time-invariant elements and an arbitrary finite number of time-varying nonlinear elements. Typically, f represents the system response and g takes into account both the independent energy sources and the initial conditions at $t = 0$. For example, (1) governs the behavior of (a) an important type of control system containing linear time-invariant elements and an arbitrary finite number of memoryless time-varying nonlinear amplifiers, or (b) an important type of electrical network containing linear time-invariant elements and an arbitrary finite number of time-varying nonlinear resistors.

The related equation

$$g(t) = f(t) + \int_{-\infty}^{\infty} k(t - \tau)\psi[f(\tau),\tau]d\tau, \qquad -\infty < t < \infty \tag{2}$$

is also often encountered. It arises when it is convenient for mathematical reasons to formulate a model of the system such that the response and excitation are defined for all $t \varepsilon (-\infty, \infty)$. In (2), usually $g \varepsilon \mathcal{L}_{\infty N}$ and only solutions belonging to $\mathcal{L}_{\infty N}$ are of interest.

One of the classic problems in the analysis of nonlinear physical systems is the determination of the properties of the response of a system, governed by an equation of the form (1), to a periodic input applied at $t = 0$. Usually, the functions $\psi_n(w,t)$, which enter into the definition of $\psi[\cdot, \cdot]$, are independent of t; g can be written as $g = g_1 + g_2$ in which $g_1 \varepsilon \mathcal{K}_N \cap \mathcal{L}_{\infty N+}$, $g_2 \varepsilon \mathcal{L}_{2N+}$, and $g_2(t) \to 0$ as $t \to \infty$; and (in accordance with the usual Volterra integral equation theory) it is known that there exists a solution $f \varepsilon \mathcal{E}_{N+}$. In a great many cases of engineering interest it is simply *assumed* that there exists a unique response and that it is ultimately periodic with the period of the input. This is a

central assumption associated, for example, with the well-known describing-function technique for the approximate determination of the steady-state response of nonlinear systems.

In connection with the actual determination of the steady-state response, two common engineering assumptions are (in effect) that there exists a unique element of $\mathfrak{L}_{\infty N} \cap \mathfrak{K}_N$, $\hat{f}$, that satisfies

$$g_1(t) = \hat{f}(t) + \int_{-\infty}^{t} k(t-\tau)\psi[\hat{f}(\tau)]d\tau, \qquad -\infty < t < \infty$$

and that the solution of (1), with $g = g_1 + g_2$, approaches $\hat{f}(t)$ as $t \to \infty$, the principal ideas evidently being that if the physical system is stable in some suitable sense, then the effect of the initial conditions at $t = 0$ should eventually "die out," and, moreover, that the steady-state response of the system should be obtained "at once" if the periodic excitation is applied at "$t = -\infty$."

The purpose of this paper is to report on some mathematical results concerning the properties of (1) and (2) that are pertinent, to a considerable extent, to engineering questions of the type discussed. In particular, as an application of our first theorem, we establish the mathematical validity of the engineering assumptions described above under what amount to reasonable conditions for the case in which $k(\cdot)$ is the matrix-valued weighting function of a passive network and $\psi[\cdot,\cdot]$ represents N positive-slope nonlinear resistors (see Theorem 3 and associated remarks).

Under similar conditions, it is proved that an equation of the type (2) possesses at most one $\mathfrak{L}_{\infty N}$ solution. This type of result is of direct interest with regard to the qualitative nature of the solutions of (2), for if our conditions are met, and, as is often the case, (a) g in (2) is periodic with period T, (b) the $\psi_n(w,t)$ are periodic in t with period T, and (c) f is an $\mathfrak{L}_{\infty N}$ solution of (2), then [since $f(t + T)$ is also a solution of (2)] it is clear that f must be periodic with period T.

III. RESULTS

Theorem 1, below, focuses attention on a relation between the solutions of (1) and (2). This theorem is later used in order to obtain conditions under which the solution of (1) approaches a periodic steady state as $t \to \infty$, when g approaches a periodic steady state as $t \to \infty$.

Theorem 1: Let

$$h_1(t) = f_1(t) + \int_{-\infty}^{t} k(t-\tau)\psi[f_1(\tau),\tau]d\tau, \qquad -\infty < t < \infty$$

$$h_2(t) = f_2(t) + \int_0^t k(t - \tau)\psi[f_2(\tau),\tau]d\tau, \qquad 0 \leqq t < \infty$$

in which $h_1 \varepsilon \mathcal{H}_N$, $f_1 \varepsilon \mathcal{H}_N \cap \mathcal{E}_{N+}$, $h_2 \varepsilon \mathcal{H}_{N+}$, *and* $f_2 \varepsilon \mathcal{E}_{N+}$. *Suppose that*

(i) $(h_1 - h_2) \varepsilon \mathcal{L}_{2N+}$

(ii) $\int_{-\infty}^0 k(t - \tau)\psi[f_1(\tau),\tau]d\tau \ \varepsilon\ \mathcal{L}_{2N+}$

and that, with

$$K(s) = \int_0^\infty k(t)e^{-st}dt \quad \text{for} \quad \sigma \geqq 0,$$

(iii) $\det [1_N + \frac{1}{2}(\alpha + \beta)K(s)] \neq 0 \quad \text{for} \quad \sigma \geqq 0$

(iv) $\frac{1}{2}(\beta - \alpha) \sup_{-\infty<\omega<\infty} \Lambda\{[1_N + \frac{1}{2}(\alpha + \beta)K(i\omega)]^{-1}K(i\omega)\} < 1.$

Then $(f_1 - f_2) \varepsilon \mathcal{L}_{2N+}$, *and, with*

$$\rho_1 = \sup_{-\infty<\omega<\infty} \Lambda\{[1_N + \tfrac{1}{2}(\alpha + \beta)K(i\omega)]^{-1}\}$$

$$\rho_2 = \sup_{-\infty<\omega<\infty} \Lambda\{[1_N + \tfrac{1}{2}(\alpha + \beta)K(i\omega)]^{-1}K(i\omega)\},$$

$$\| f_1 - f_2 \|_+ \leqq \rho_1[1 - \tfrac{1}{2}(\beta - \alpha)\rho_2]^{-1} \cdot \left\| h_1 - h_2 - \int_{-\infty}^0 k(t - \tau)\psi[f_1(\tau),\tau]d\tau \right\|_+ .$$

If, in addition to the hypotheses stated above,

$$h_1(t) - h_2(t) - \int_{-\infty}^0 k(t - \tau)\psi[f_1(\tau),\tau]d\tau \to 0$$

as $t \to \infty$, *and*

$$\int_0^\infty | k_{mn}(t) |^2 dt < \infty \qquad (m,n = 1, 2, \cdots, N),$$

then $[f_1(t) - f_2(t)] \to 0$ *as* $t \to \infty$.

Our next result is concerned with the character of the change in the solution of (2) when g is altered by the addition of an element of $\mathcal{L}_{2N}$.

Theorem 2: Let

$$h_1(t) = f_1(t) + \int_{-\infty}^\infty k(t - \tau)\psi[f_1(\tau),\tau]d\tau, \qquad -\infty < t < \infty$$

$$h_2(t) = f_2(t) + \int_{-\infty}^\infty k(t - \tau)\psi[f_2(\tau),\tau]d\tau, \qquad -\infty < t < \infty$$

in which: $h_1, h_2 \varepsilon \mathcal{K}_N$; $f_1, f_2 \varepsilon \mathcal{L}_{\infty N}$; *and* $(h_1 - h_2) \varepsilon \mathcal{L}_{2N}$. *Suppose that*

(*i*)

$$\int_0^\infty \left| \int_t^\infty | k_{mn}(x) | dx \right|^2 dt + \int_{-\infty}^0 \left| \int_{-\infty}^t | k_{mn}(x) | dx \right|^2 dt < \infty, \qquad (m,n = 1, 2, \cdots, N)$$

and that, with

$$K(i\omega) = \int_{-\infty}^\infty k(t) e^{-i\omega t} dt,$$

(*ii*) $\det [1_N + \frac{1}{2}(\alpha + \beta)K(i\omega)] \neq 0$ *for all* ω

(*iii*) $\frac{1}{2}(\beta - \alpha) \sup_{-\infty<\omega<\infty} \Lambda\{[1_N + \frac{1}{2}(\alpha + \beta)K(i\omega)]^{-1}K(i\omega)\} < 1.$

Then $(f_1 - f_2) \varepsilon \mathcal{L}_{2N}$, *and, with*

$$\rho_1 = \sup_{-\infty<\omega<\infty} \Lambda\{[1_N + \tfrac{1}{2}(\alpha + \beta)K(i\omega)]^{-1}\}$$

$$\rho_2 = \sup_{-\infty<\omega<\infty} \Lambda\{[1_N + \tfrac{1}{2}(\alpha + \beta)K(i\omega)]^{-1}K(i\omega)\},$$

$$\| f_1 - f_2 \| \leqq \rho_1[1 - \tfrac{1}{2}(\beta - \alpha)\rho_2]^{-1} \| h_1 - h_2 \|.$$

Observe that Theorem 2 implies that if (*i*), (*ii*) and (*iii*) are satisfied, then (2) possesses at most one $\mathcal{L}_{\infty N}$ solution.

As indicated earlier, in many cases of engineering interest g, in (1), can be written as $g = g_1 + g_2$, in which $g_1 \varepsilon \mathcal{K}_N \cap \mathcal{L}_{\infty N+}$, $g_2 \varepsilon \mathcal{L}_{2N+}$, and $g_2(t) \to 0$ as $t \to \infty$. In such cases it is often of considerable importance to determine whether $f(t)$ approaches a steady-state response that is periodic with period T as $t \to \infty$. As a specific application of Theorem 1, the following result is proved.

Theorem 3: Let $g_1 \varepsilon \mathcal{K}_N \cap \mathcal{L}_{\infty N+}$, $g_2 \varepsilon \mathcal{L}_{2N+}$, $g_2(t) \to 0$ *as* $t \to \infty$, $\psi_n(w,t) = \psi_n(w,t + T)$ *for all* w *and* t *and* $n = 1, 2, \cdots, N$, *and* $\psi[0,t] \varepsilon \mathcal{K}_N$. *Let* $f \varepsilon \mathcal{E}_{N+}$ *satisfy*

$$g_1(t) + g_2(t) = f(t) + \int_0^t k(t - \tau)\psi[f(\tau),\tau]\, d\tau, \qquad 0 \leqq t < \infty.$$

Suppose that

(*i*) $\displaystyle\int_0^\infty \left| \int_t^\infty | k_{mn}(x) | dx \right|^2 dt < \infty \qquad (m, n = 1, 2, \cdots, N)$

(*ii*) $\displaystyle\int_0^\infty | (1 + t) k_{mn}(t) |^2 dt < \infty \qquad (m, n = 1, 2, \cdots, N)$

and that, with

$$K(s) = \int_0^\infty k(t)e^{-st}dt \quad \textit{for} \quad \sigma \geqq 0,$$

(*iii*) $\det\,[1_N + \frac{1}{2}(\alpha + \beta)K(s)] \neq 0 \quad \textit{for} \quad \sigma \geqq 0$

(*iv*) $\frac{1}{2}(\beta - \alpha) \sup_{-\infty<\omega<\infty} \Lambda\{[1_N + \frac{1}{2}(\alpha + \beta)K(i\omega)]^{-1}K(i\omega)\} < 1.$

Then there exists a unique $\hat{f} \,\varepsilon\, \mathcal{K}_N$ *such that*

$$g_1(t) = \hat{f}(t) + \int_{-\infty}^{t} k(t - \tau)\psi[\hat{f}(\tau),\tau]d\tau, \qquad -\infty < t < \infty.$$

Moreover, $\hat{f} \,\varepsilon\, \mathcal{L}_{\infty N}$, $(f - \hat{f}) \,\varepsilon\, \mathcal{L}_{2N+}$, *and*

$$[f(t) - \hat{f}(t)] \rightarrow 0 \quad as \quad t \rightarrow \infty.$$

With regard to the hypotheses of Theorems 1 and 3, it can be shown that*

$$\det\,[1_N + \tfrac{1}{2}(\alpha + \beta)K(s)] \neq 0 \quad \text{for} \quad \sigma \geqq 0$$

and

$$\tfrac{1}{2}(\beta - \alpha) \sup_{-\infty<\omega<\infty} \Lambda\{[1_N + \tfrac{1}{2}(\alpha + \beta)K(i\omega)]^{-1}K(i\omega)\} < 1$$

provided that $\alpha \geqq 0$ and $[K(i\omega) + K(i\omega)^*]$ is nonnegative definite for all ω. For this reason our results are particularly relevant to the theory of passive nonlinear electrical networks.

IV. PROOFS

4.1 *Proof of Theorem 1*

Let $\mathbf{K}$ denote the bounded linear mapping of $\mathcal{L}_{2N+}$ into itself defined by

$$\mathbf{K}f = \int_0^t k(t - \tau)f(\tau)d\tau, \qquad f \,\varepsilon\, \mathcal{L}_{2N+}.$$

With y an arbitrary positive number, and f an arbitrary element of $\mathcal{K}_{N+}$, let $\mathbf{P}$ denote the mapping of $\mathcal{K}_{N+}$ into itself defined by $\mathbf{P}f = f_y$, and let ψf denote the N-vector-valued function of t with values

$$\psi[f(t),t] \quad \text{for} \quad 0 \leqq t < \infty.$$

* The validity of the first assertion can be established with a standard argument involving the analyticity of $K(s)$ for $\sigma > 0$. The second statement is a direct extension of a result proved in Ref. 1. In particular, the greatest lower bound (over n) of the smallest eigenvalue of the term $[1_N + R_n]^{-1*}\,[1_N + R_n + R_n^*][1_N + R_n]^{-1}$, which appears in (7) of Ref. 1, can easily be shown to be positive. Thus, the conclusion of Theorem 2 of Ref. 1 remains valid if the condition $\alpha > 0$ is replaced by $\alpha \geqq 0$.

Then from

$$h_1(t) - h_2(t) - \int_{-\infty}^{0} k(t-\tau)\psi[f_1(\tau),\tau]d\tau$$
$$= f_1(t) - f_2(t) + \int_0^t k(t-\tau)(\psi[f_1(\tau),\tau] - \psi[f_2(\tau),\tau])d\tau, \qquad (3)$$
$$0 \leqq t < \infty$$

and the fact that

$$\mathbf{P}\int_0^t k(t-\tau)(\psi[f_1(\tau),\tau] - \psi[f_2(\tau),\tau])d\tau$$
$$= \mathbf{P}\int_0^t k(t-\tau)(\psi[f_{1y}(\tau),\tau] - \psi[f_{2y}(\tau),\tau])d\tau,$$

we obtain

$$h_y = \mathbf{P}[\mathbf{I} + \tfrac{1}{2}(\alpha+\beta)\mathbf{K}](f_{1y} - f_{2y})$$
$$+ \mathbf{PK}\{\psi f_{1y} - \psi f_{2y} - \tfrac{1}{2}(\alpha+\beta)(f_{1y} - f_{2y})\} \qquad (4)$$

in which $\mathbf{I}$ denotes the identity operator on $\mathfrak{L}_{2N+}$ and

$$h_y = h_{1y} - h_{2y} - \mathbf{P}\int_{-\infty}^{0} k(t-\tau)\psi[f_1(\tau),\tau]d\tau.$$

In order to proceed we need the following result.[2]

Lemma 1: Let $\det[1_N + \frac{1}{2}(\alpha+\beta)K(s)] \neq 0$ *for* $\sigma \geqq 0$. *Then* $[\mathbf{I} + \frac{1}{2}(\alpha+\beta)\mathbf{K}]$ *possesses a bounded inverse on* $\mathfrak{L}_{2N+}$, *and*

$$\| [\mathbf{I} + \tfrac{1}{2}(\alpha+\beta)\mathbf{K}]^{-1} \|_+ \leqq \sup_{-\infty<\omega<\infty} \Lambda\{[1_N + \tfrac{1}{2}(\alpha+\beta)K(i\omega)]^{-1}\}$$

$$\| [\mathbf{I} + \tfrac{1}{2}(\alpha+\beta)\mathbf{K}]^{-1}\mathbf{K} \|_+ \leqq \sup_{-\infty<\omega<\infty} \Lambda\{[1_N + \tfrac{1}{2}(\alpha+\beta)K(i\omega)]^{-1}K(i\omega)\}.$$

Furthermore,

$$\mathbf{P}[\mathbf{I} + \tfrac{1}{2}(\alpha+\beta)\mathbf{K}]^{-1} = \mathbf{P}[\mathbf{I} + \tfrac{1}{2}(\alpha+\beta)\mathbf{K}]^{-1}\mathbf{P} \quad \textit{for all} \quad y > 0.$$

Thus, since

$$\mathbf{P}[\mathbf{I} + \tfrac{1}{2}(\alpha+\beta)\mathbf{K}]^{-1}\mathbf{P}[\mathbf{I} + \tfrac{1}{2}(\alpha+\beta)\mathbf{K}](f_{1y} - f_{2y}) = f_{1y} - f_{2y},$$

we obtain from (4)

$$f_{1y} - f_{2y} = \mathbf{P}[\mathbf{I} + \tfrac{1}{2}(\alpha+\beta)\mathbf{K}]^{-1}h_y$$
$$- \mathbf{P}[\mathbf{I} + \tfrac{1}{2}(\alpha+\beta)\mathbf{K}]^{-1}\mathbf{PK}\{\psi f_{1y} - \psi f_{2y} - \tfrac{1}{2}(\alpha+\beta)(f_{1y} - f_{2y})\}.$$

Using the fact that

$$\| \psi f_{1y} - \psi f_{2y} - \tfrac{1}{2}(\alpha + \beta)(f_{1y} - f_{2y}) \|_+ \leqq \tfrac{1}{2}(\beta - \alpha) \| f_{1y} - f_{2y} \|_+ ,$$

it follows that

$$\begin{aligned} \| f_{1y} - f_{2y} \|_+ \leqq{} & \| \mathbf{P}[\mathbf{I} + \tfrac{1}{2}(\alpha + \beta)\mathbf{K}]^{-1} h_y \|_+ \\ & + \tfrac{1}{2}(\beta - \alpha) \| \mathbf{P}[\mathbf{I} + \tfrac{1}{2}(\alpha + \beta)\mathbf{K}]^{-1}\mathbf{PK} \|_+ \\ & \cdot \| f_{1y} - f_{2y} \|_+ \\ \leqq{} & \| [\mathbf{I} + \tfrac{1}{2}(\alpha + \beta)\mathbf{K}]^{-1} \|_+ \| h_y \|_+ \\ & + \tfrac{1}{2}(\beta - \alpha) \| [\mathbf{I} + \tfrac{1}{2}(\alpha + \beta)\mathbf{K}]^{-1}\mathbf{K} \|_+ \\ & \cdot \| f_{1y} - f_{2y} \|_+ . \end{aligned}$$

Using the inequalities of the lemma,

$$\begin{aligned} \| f_{1y} - f_{2y} \|_+ & \leqq \rho_1[1 - \tfrac{1}{2}(\beta - \alpha)\rho_2]^{-1} \| h_y \|_+ \\ & \leqq \rho_1[1 - \tfrac{1}{2}(\beta - \alpha)\rho_2]^{-1} \\ & \quad \cdot \left\| h_1 - h_2 - \int_{-\infty}^{0} k(t - \tau)\psi[f_1(\tau),\tau]d\tau \right\|_+ \end{aligned} \tag{5}$$

for *all* $y > 0$. Therefore, $(f_1 - f_2)\ \varepsilon\ \mathfrak{L}_{2N+}$ and $\| f_1 - f_2 \|_+$ possesses the upper bound stated in the theorem.

We now show that $(f_1 - f_2)\ \varepsilon\ \mathfrak{L}_{2N+}$,

$$h_1(t) - h_2(t) - \int_{-\infty}^{0} k(t - \tau)\psi[f_1(\tau),\tau]d\tau \to 0 \quad \text{as} \quad t \to \infty , \tag{6}$$

and

$$\int_0^\infty | k_{mn}(t) |^2 \, dt < \infty \qquad (m, n = 1, 2, \cdots, N) \tag{7}$$

imply that $[f_1(t) - f_2(t)] \to 0$ as $t \to \infty$.

Assume that $(f_1 - f_2)\ \varepsilon\ \mathfrak{L}_{2N+}$ and that (6) and (7) hold. Then, from (3) it is evident that $[f_1(t) - f_2(t)] \to 0$ as $t \to \infty$ if

$$\int_0^t k(t - \tau)(\psi[f_1(\tau),\tau] - \psi[f_2(\tau),\tau])d\tau \to 0 \quad \text{as} \quad t \to \infty . \tag{8}$$

To prove that (8) is satisfied, observe first that $(f_1 - f_2)\ \varepsilon\ \mathfrak{L}_{2N+}$ implies that $(\psi f_1 - \psi f_2)\ \varepsilon\ \mathfrak{L}_{2N+}$. Thus it suffices to show that if $g\ \varepsilon\ \mathfrak{L}_{2N+}$, then

$$\int_0^t k(t - \tau)g(\tau)d\tau \to 0 \quad \text{as} \quad t \to \infty .$$

Let

$$G(i\omega) = \text{l.i.m.} \int_0^\infty g(t)e^{-i\omega t}dt, \qquad g \in \mathfrak{L}_{2N+}.$$

Then, in view of assumption (7), the modulus of any element of the N-vector $K(i\omega)G(i\omega)$ is integrable on the ω-set $(-\infty, \infty)$, and hence by the Riemann-Lebesgue lemma

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} K(i\omega)G(i\omega)e^{i\omega t}d\omega,$$

which is equal to

$$\int_0^t k(t-\tau)g(\tau)d\tau,$$

approaches zero as $t \to \infty$. This completes the proof of Theorem 1.

4.2 *Proof of Theorem 2*

In this section, $\mathbf{K}$ denotes the bounded linear mapping of $\mathfrak{L}_{2N}$ into itself defined by

$$\mathbf{K}f = \int_{-\infty}^{\infty} k(t-\tau)f(\tau)d\tau, \qquad f \in \mathfrak{L}_{2N}.$$

With y an arbitrary positive number and f an arbitrary element of $\mathcal{H}_N$, $\mathbf{P}$ denotes the mapping of $\mathcal{H}_N$ into itself defined by $\mathbf{P}f = f_y$, and ψf denotes the N-vector-valued function of t with values

$$\psi[f(t),t] \quad \text{for} \quad -\infty < t < \infty.$$

From the fact that

$$\begin{aligned} h_1(t) - h_2(t) &= f_1(t) - f_2(t) \\ &\qquad + \int_{-\infty}^{\infty} k(t-\tau)(\psi[f_1(\tau),\tau] - \psi[f_2(\tau),\tau])d\tau, \end{aligned} \tag{9}$$

we obtain

$$\begin{aligned} h_y &= f_{1y} - f_{2y} + \mathbf{K}(\psi f_{1y} - \psi f_{2y}) \\ &= [\mathbf{I} + \tfrac{1}{2}(\alpha+\beta)\mathbf{K}](f_{1y} - f_{2y}) \end{aligned} \tag{10}$$

$$\qquad + \mathbf{K}\{\psi f_{1y} - \psi f_{2y} - \tfrac{1}{2}(\alpha+\beta)(f_{1y} - f_{2y})\}, \tag{11}$$

in which $\mathbf{I}$ denotes the identity operator on $\mathfrak{L}_{2N}$, and

$$h_y(t) = h_{1y}(t) - h_{2y}(t) + \int_{-\infty}^{\infty} k(t - \tau)(\psi[f_{1y}(\tau),\tau] - \psi[f_{2y}(\tau),\tau])d\tau$$
$$- \mathbf{P}\int_{-\infty}^{\infty} k(t - \tau)(\psi[f_1(\tau),\tau] - \psi[f_2(\tau),\tau])d\tau .$$

At this point we need[2,†]

Lemma 2: If $\det [1_N + \frac{1}{2}(\alpha + \beta)K(i\omega)] \neq 0$ *for all* ω, *then*

$$[\mathbf{I} + \tfrac{1}{2}(\alpha + \beta)\mathbf{K}]$$

possesses a bounded inverse on $\mathcal{L}_{2N}$, *and*

$$\| [\mathbf{I} + \tfrac{1}{2}(\alpha + \beta)\mathbf{K}]^{-1} \| \leqq \sup_{-\infty<\omega<\infty} \Lambda\{[1_N + \tfrac{1}{2}(\alpha + \beta)K(i\omega)]^{-1}\}$$

$$\| [\mathbf{I} + \tfrac{1}{2}(\alpha + \beta)\mathbf{K}]^{-1}\mathbf{K} \| \leqq \sup_{-\infty<\omega<\infty} \Lambda\{[1_N + \tfrac{1}{2}(\alpha + \beta)K(i\omega)]^{-1}K(i\omega)\}.$$

Thus from (11),

$$f_{1y} - f_{2y}$$
$$= -[\mathbf{I} + \tfrac{1}{2}(\alpha + \beta)\mathbf{K}]^{-1}\mathbf{K}\{\psi f_{1y} - \psi f_{2y} - \tfrac{1}{2}(\alpha + \beta)(f_{1y} - f_{2y})\}$$
$$+ [\mathbf{I} + \tfrac{1}{2}(\alpha + \beta)\mathbf{K}]^{-1}h_y .$$

Using the fact that

$$\| \psi f_{1y} - \psi f_{2y} - \tfrac{1}{2}(\alpha + \beta)(f_{1y} - f_{2y}) \| \leqq \tfrac{1}{2}(\beta - \alpha) \| f_{1y} - f_{2y} \|,$$

we have

$$\| f_{1y} - f_{2y} \| \leqq \tfrac{1}{2}(\beta - \alpha) \| [\mathbf{I} + \tfrac{1}{2}(\alpha + \beta)\mathbf{K}]^{-1}\mathbf{K} \| \cdot \| f_{1y} - f_{2y} \|$$
$$+ \| [\mathbf{I} + \tfrac{1}{2}(\alpha + \beta)\mathbf{K}]^{-1} \| \cdot \| h_y \|.$$

In view of the inequalities of the lemma,

$$\| f_{1y} - f_{2y} \| \leqq \rho_1[1 - \tfrac{1}{2}(\beta - \alpha)\rho_2]^{-1} \| h_y \|. \tag{12}$$

Assume now that there exists a constant c such that $\| h_y \| \leqq c$ for all $y > 0$. Then, from (12), it is clear that $(f_1 - f_2) \varepsilon \mathcal{L}_{2N}$. This implies that $(\psi f_1 - \psi f_2) \varepsilon \mathcal{L}_{2N}$. Hence, (9) can be written as

$$h_1 - h_2 = f_1 - f_2 + \mathbf{K}(\psi f_1 - \psi f_2),$$

from which it follows, by essentially the same argument as that used to obtain (12) from (10), that

† With no more than a reinterpretation of the functions involved, the proofs of the inequalities of Lemma 1 suffice to establish the inequalities of Lemma 2.

$$\| f_1 - f_2 \| \leqq \rho_1[1 - \tfrac{1}{2}(\beta - \alpha)\rho_2]^{-1} \| h_1 - h_2 \|.$$

Therefore to complete the proof of Theorem 2, it suffices to prove

Lemma 3: If $(h_1 - h_2) \,\varepsilon\, \mathfrak{L}_{2N}$, $f_1, f_2 \,\varepsilon\, \mathfrak{L}_{\infty N}$, *and assumption (i) of Theorem 2 is satisfied, then there exists a constant c such that* $\| h_y \| \leqq c$ *for all* $y > 0$.

4.2.1 *Proof of Lemma 3*

Let $q = (q_1, q_2, \cdots, q_N)' = (\psi f_1 - \psi f_2)$,

$$\begin{aligned} \theta(t) &= 1 \quad \text{for} \quad |t| \leqq y \\ &= 0 \quad \text{for} \quad |t| > y, \end{aligned}$$

and

$$u = (u_1, u_2, \cdots, u_N)' = \int_{-\infty}^{\infty} k(t - \tau)[\theta(\tau) - \theta(t)]q(\tau)d\tau.$$

Then, since $(h_1 - h_2) \,\varepsilon\, \mathfrak{L}_{2N}$, it is sufficient to prove that there exists a constant c_1 such that $\| u \| \leqq c_1$ for all $y > 0$. Further, since

$$\begin{aligned} \| u \|^2 &= \sum_{m=1}^{N} \int_{-\infty}^{\infty} | u_m(t) |^2 \, dt \\ &= \sum_{m=1}^{N} \int_{-\infty}^{\infty} \left| \sum_{n=1}^{N} \int_{-\infty}^{\infty} k_{mn}(t - \tau)[\theta(\tau) - \theta(t)]q_n(\tau)d\tau \right|^2 dt \\ &\leqq N \sum_{m=1}^{N} \sum_{n=1}^{N} \int_{-\infty}^{\infty} \left| \int_{-\infty}^{\infty} k_{mn}(t - \tau)[\theta(\tau) - \theta(t)]q_n(\tau)d\tau \right|^2 dt \\ &\leqq \eta N \sum_{m=1}^{N} \sum_{n=1}^{N} \int_{-\infty}^{\infty} \left| \int_{-\infty}^{\infty} | k_{mn}(t - \tau) | \cdot | \theta(\tau) - \theta(t) | \, d\tau \right|^2 dt, \end{aligned}$$

in which

$$\eta = \max_n \sup_t | q_n(t) |^2,$$

it suffices to show that there exists a constant c_2 such that for all $y > 0$

$$\int_{-\infty}^{\infty} \left| \int_{-\infty}^{\infty} | k_{mn}(t - \tau) | \cdot | \theta(\tau) - \theta(t) | \, d\tau \right|^2 dt \leqq c_2$$

$$(m, n = 1, 2, \cdots, N).$$

Using the fact that

$$\int_{-\infty}^{\infty} | k_{mn}(\tau) | \, | \theta(t - \tau) - \theta(t) | \, d\tau$$

$$= \int_{t+y}^{\infty} | k_{mn}(\tau) | \, d\tau + \int_{-\infty}^{t-y} | k_{mn}(\tau) | \, d\tau \quad \text{for} \quad | t | \leqq y$$

$$= \int_{t-y}^{t+y} | k_{mn}(\tau) | \, d\tau \qquad \text{for} \quad | t | > y,$$

it is a simple matter to verify that

$$\int_{-\infty}^{\infty} \left| \int_{-\infty}^{\infty} | k_{mn}(t - \tau) | \, | \theta(\tau) - \theta(t) | \, d\tau \right|^2 dt$$

$$\leqq 2 \int_0^{2y} \left| \int_x^{\infty} | k_{mn}(\tau) | \, d\tau \right|^2 dx + 2 \int_{-2y}^{0} \left| \int_{-\infty}^{x} | k_{mn}(\tau) | \, d\tau \right|^2 dx$$

$$+ \int_{-\infty}^{0} \left| \int_{x-2y}^{x} | k_{mn}(\tau) | \, d\tau \right|^2 dx$$

$$+ \int_0^{\infty} \left| \int_x^{x+2y} | k_{mn}(\tau) | \, d\tau \right|^2 dx,$$

from which it is evident that our assumptions imply that there exists a c_2 with the required property. This proves the lemma and completes the proof of Theorem 2.

Remark:

Assumption (i) of Theorem 2 is satisfied if

$$\int_{-\infty}^{\infty} | t k_{mn}(t) | \, dt < \infty, \qquad (n, m = 1, 2, \cdots, N),$$

for then the (bounded) functions

$$\int_t^{\infty} | k_{mn}(x) | \, dx \quad \text{and} \quad \int_{-\infty}^{t} | k_{mn}(x) | \, dx$$

are integrable on $(0, \infty)$ and $(-\infty, 0)$, respectively.

4.3 *Proof of Theorem 3*

We need two lemmas.

Lemma 4: Let $\psi[\cdot\,,\cdot]$ satisfy the conditions of Theorem 3, $g_1 \in \mathcal{K}_N$, and

$$K(i\omega) = \int_0^{\infty} k(t) e^{-i\omega t} dt.$$

Suppose that, with $\mathfrak{N}$ the set of integers,

$$(i)\ det\left[1_N + \tfrac{1}{2}(\alpha+\beta)K\left(\frac{i2\pi n}{T}\right)\right] \neq 0 \quad for \quad n\ \varepsilon\ \mathfrak{N}$$

$$(ii)\ \tfrac{1}{2}(\beta-\alpha)\sup_{n\varepsilon\mathfrak{N}} \Lambda\left\{\left[1_N + \tfrac{1}{2}(\alpha+\beta)K\left(\frac{i2\pi n}{T}\right)\right]^{-1} K\left(\frac{i2\pi n}{T}\right)\right\} < 1.$$

Then there exists a unique $\hat{f}\ \varepsilon\ \mathcal{K}_N$ such that

$$g_1(t) = \hat{f}(t) + \int_{-\infty}^{t} k(t-\tau)\psi[\hat{f}(\tau),\tau]d\tau, \qquad -\infty < t < \infty.$$

Proof of Lemma 4:

Theorem 4 of Ref. 1 and the remarks relating to its proof imply that the conclusion of Lemma 4 is valid if the hypotheses of the lemma are satisfied and the condition

$$\sup_{n\varepsilon\mathfrak{N}} \Lambda\left\{\left[1_N + \tfrac{1}{2}(\alpha+\beta)K\left(\frac{i2\pi n}{T}\right)\right]^{-1}\right\} < \infty \tag{13}$$

is met. However, since every element of $K(i2\pi n/T)$ approaches zero as $|n| \to \infty$, assumption (i) of Lemma 1 implies that

$$\inf_{n\varepsilon\mathfrak{N}}\left|\det\left[1_N + \tfrac{1}{2}(\alpha+\beta)K\left(\frac{i2\pi n}{T}\right)\right]\right| > 0.$$

Therefore, in view of the fact that the elements of $K(i2\pi n/T)$ are uniformly bounded for $n\ \varepsilon\ \mathfrak{N}$, it follows that (13) is satisfied. This proves the lemma.

Lemma 5: Let $\psi[\cdot,\cdot]$ satisfy the conditions of Theorem 3, let $\hat{f}\ \varepsilon\ \mathcal{K}_N$, and suppose that assumption (ii) of Theorem 3 is satisfied. Then

$$\int_{-\infty}^{t} k(t-\tau)\psi[\hat{f}(\tau),\tau]d\tau\ \varepsilon\ \mathfrak{L}_{\infty N}.$$

Proof of Lemma 5:

Let $q(t) = \psi[\hat{f}(t),t]$, and

$$u = (u_1, u_2, \cdots, u_N)' = \int_{-\infty}^{t} k(t-\tau)q(\tau)d\tau.$$

Then $q\ \varepsilon\ \mathcal{K}_N$, and

$$
\begin{aligned}
| u_m(t) | &\leqq \sum_{m=1}^{N} \int_{-\infty}^{t} | k_{mn}(t - \tau) | \cdot | q_n(\tau) | \, d\tau \\
&\leqq \sum_{m=1}^{N} \int_{0}^{\infty} | k_{mn}(\tau) | \cdot | q_n(t - \tau) | \, d\tau.
\end{aligned}
$$

Furthermore,

$$
\left| \int_0^\infty | k_{mn}(\tau) | \; | q_n(t - \tau) | \, d\tau \right|^2
$$

$$
\leqq \int_0^\infty [(1 + \tau) k_{mn}(\tau)]^2 d\tau \int_0^\infty \left| \frac{q_n(t - \tau)}{1 + \tau} \right|^2 d\tau,
$$

and the last integral can be bounded as follows

$$
\begin{aligned}
\int_0^\infty \left| \frac{q_n(t - \tau)}{1 + \tau} \right|^2 d\tau &= \sum_{m=0}^{\infty} \int_{mT}^{(m+1)T} \left| \frac{q_n(t - \tau)}{1 + \tau} \right|^2 d\tau \\
&\leqq \left(1 + \sum_{n=1}^{\infty} (mT)^{-2} \right) \int_0^T | q_n(t) |^2 \, dt.
\end{aligned}
$$

Thus, the $u_m(t)$ are uniformly bounded on $(-\infty, \infty)$, which proves the lemma.

Theorem 3 follows at once from Lemmas 4 and 5, Theorem 1, and the fact that assumption (i) of Theorem 3 and $\hat{f} \, \varepsilon \, \mathfrak{L}_{\infty N}$ imply that

$$
\int_{-\infty}^{0} k(t - \tau) \psi[\hat{f}(\tau), \tau] d\tau \; \varepsilon \; \mathfrak{L}_{2N+}.
$$

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